

Cavity Quantum Electrodynamics of a two-level atom with modulated fields

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We studied the interaction of a two-level atom with a frequency modulated cavity mode in an ideal optical cavity. The system, described by a Jaynes-Cummings Hamiltonian, gave rise to a set of stiff nonlinear first order equations solved numerically using implicit and semi-implicit numerical algorithms. We explored the evolution of the atomic system using nonlinear dynamics tools, like time series, phase plane, power spectral density, and Poincaré sections plots, for monochromatic and bichromatic modulations of the cavity field. The system showed quasiperiodic and possibly chaotic behavior when the selected monochromatic frequencies, or ratio of the bichromatic frequencies were irrational (incommensurate) numbers. In addition, when the modulated frequencies were overtones of the Rabi frequency of the system, a single dominant frequency emerged for the system.

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I. INTRODUCTION

There has been a considerable amount of interest in the study of simple quantum mechanical systems, such as the simple harmonic oscillator or the two-level atom interacting with quasiperiodic electromagnetic fields. Initially, attention was directed to understand quantum chaos phenomenon.¹⁻³ Later on, it was observed that the atomic interaction with quasiperiodic fields could result into some other novel dynamical features like power broadening and resonance shifts,⁴ multiple resonance spectral structures,⁵ population trapping,⁶ sinusoidal and square wave oscillation of the population,^{7,8} and new types of dynamical localization.⁹ It was also noted that the atomic response to quasiperiodic fields is extremely complex, and the diagnostic techniques employed for characterizing chaos could easily be utilized for studying the complexities of such interactions.

A good number of studies are available in which investigations of atomic systems interacting with a quasiperiodic amplitude modulated field have been carried out, finding that the atomic density matrix elements have rapidly decaying autocorrelation functions.¹⁰ In one of these works,¹¹ the dynamics of a two-level atom interacting with a quasiperiodic field was examined as a function of the field strength (which could be considered as a controlling parameter). Also, the atomic dynamics was characterized through the computation of Poincaré sections of the Bloch vectors's motion, and the attractor's fractal nature.¹² Another interesting work involved the situation in which a single two-level atom was interacting with a combination of a classical and a quantized fields in the cavity, so that the cavity field acted as a quantum probe of the dressed states defined by the classical laser field.¹³ Many interesting effects such as two-photon gain, cavity perturbed resonance fluorescence spectra, and the atomic squeezing in the cavity were examined. Chaotic Rabi oscillations under quasiperiodic perturbation were also

studied,¹⁴ where a system consisting of a two-level atom was coupled through a time-dependent, quasiperiodic perturbation with two incommensurate frequencies. In this system, chaos was predicted by observing a rapidly decreasing autocorrelation and by looking at the continuous Fourier spectrum of the time-dependent physical observable. More recently, the use of time-dependent dissipative environments to control the quantum state of a two-level atom was demonstrated.¹⁵ For this case, the possibility to decouple dynamically the atom from its environment leading to Markovian dynamics was studied.

In this paper, we show an alternative approach to explore the response of an atomic system to a quasiperiodic field through its interaction with a modulated single mode field sustained in an ideal cavity. The frequency of the cavity field is modulated while its amplitude is kept constant, making this work different from the studies reported earlier. The interaction of a modulated field with a two-level atom is analyzed by studying the dynamical evolution, phase plot, Poincaré section, and power spectrum density of the system under certain parametric conditions. We solve the set of nonlinear differential equations that describe the evolution of the system using implicit and semi-implicit algorithms specialized in dealing with stiff differential equations. We observe that for specific frequency values, the system's evolution is quasiperiodic, and the possibility of chaos is present. The cavity field is first modulated monochromatically, and secondly bichromatically such that the frequencies involved and their ratios are either irrational numbers or integral multiples of the Rabi frequency of the system. The paper is organized as follows. In section II, we describe the physical model of a two-level atom interacting with a modulated quantized cavity-field mode. Section III is devoted to results and discussions of the studied atomic system. Finally, some concluding remarks are presented in section IV.

II. MODEL AND BASIC EQUATIONS

We consider the interaction of a two-level atom at rest, and atomic transition frequency ω_0 , with a quasiperiodic quantized cavity field $\omega_c(t)$. The quasiperiodic (or incommensurate) field is obtained by modulating the quantized cavity field frequency. The Hamiltonian of the system under the rotating wave approximation is

$$H = \omega_0 S_z + \omega_c(t) a^\dagger a + g(S_+ a + a^\dagger S_-), \quad (1)$$

where $\hbar = 1$ for simplicity, a and a^\dagger are the annihilation and creation operators for the cavity field, and g is the atom field coupling coefficient; S_z and S_\pm are the usual spin- $\frac{1}{2}$ operators satisfying the commutation relations

$$[S_z, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = 2S_z. \quad (2)$$

If the modulation of the cavity field is slow, that is, the variation of $\omega_c(t)$ with respect to ω_0 is small, then the operator $a^\dagger a + S_z$ is a constant of motion, and the *interaction* picture Hamiltonian of the system takes the form

$$H_I = \delta(t) S_z + g(S_+ a + a^\dagger S_-), \quad (3)$$

where $\delta(t) = \omega_0 - \omega_c(t)$. In this picture, the operators incorporate a dependency on time, and the equation of motion for an arbitrary time-dependent operator q is given by

$$i\dot{q} = [q, H_I]. \quad (4)$$

Hence the Heisenberg's equations of motion for the atomic and field operators of Hamiltonian (3) are

$$\begin{aligned} i\dot{a} &= gS_-, \\ i\dot{S}_z &= g(S_+ a - a^\dagger S_-), \\ i\dot{S}_+ &= -\delta S_+ + 2ga^\dagger S_z, \\ i\dot{S}_- &= \delta S_- - 2gaS_z. \end{aligned} \quad (5)$$

Equation (5) gives rise to a *nonlinear* third order differential equation in S_z

$$\begin{aligned} \ddot{S}_z - \frac{\dot{\delta}}{\delta} \dot{S}_z + [\delta^2 + 4g^2(N + 1/2)] \dot{S}_z \\ - 4g^2(N + 1/2) \frac{\dot{\delta}}{\delta} S_z = 0, \end{aligned} \quad (6)$$

where $N \equiv a^\dagger a + S_z$ is a constant of motion.

If we define the following new variables

$$\begin{aligned} y_1 &= \langle S_z \rangle, \\ y_2 &= \dot{y}_1 = \langle \dot{S}_z \rangle, \\ y_3 &= \dot{y}_2 = \langle \ddot{S}_z \rangle, \end{aligned} \quad (7)$$

then it is possible to recast (6) into the equivalent system

of three, nonlinear first order differential equations

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= y_3, \\ \dot{y}_3 &= \frac{\dot{\delta}}{\delta} y_3 - [\delta^2 + 4g^2(N + 1/2)] y_2 \\ &\quad + 4g^2(N + 1/2) \frac{\dot{\delta}}{\delta} y_1. \end{aligned} \quad (8)$$

Solving either (5) or (8) numerically, the dynamical evolution of the system can easily be obtained for different parametric conditions.

The system of nonlinear differential equations (8) includes some terms ($\dot{\delta}/\delta$ and δ^2) that lead to rapid oscillations of the solution. This makes the set of equations *stiff*, so very stable numerical algorithms must be used to find its solutions. Because this system was small in size (only three differential equations) and we looked for solutions with moderate accuracies, we used *implicit* Runge-Kutta, and *semi-implicit* Rosenbrock methods to solve the system.¹⁶

We estimated the *power spectral density* (PSD) for each one of the studied cases to identify periodicities, dominant frequencies, and their correlation with the Rabi frequency Ω of the system. To obtain these power densities, we used a simple *periodogram* estimator by taking an n -point sample of the function $\langle S_z(t) \rangle$ at equal intervals Δ , and using the fast Fourier transform to compute its *forward* discrete Fourier transform¹⁷

$$Y_k = \sum_{j=0}^{n-1} \langle S_z \rangle_j e^{-2\pi j k \sqrt{-1}/n} \quad k = 0, 1, 2, \dots \quad (9)$$

Then the power spectrum was defined at $n/2 + 1$ frequencies as

$$\begin{aligned} P(f_k) &= \frac{1}{n^2} [|Y_k|^2 + |Y_{n-k}|^2] \\ k &= 1, 2, \dots, \left(\frac{n}{2} - 1\right), \end{aligned} \quad (10)$$

where f_k took values only for zero and positive frequencies

$$f_k = \frac{\omega_k}{2\pi} = \frac{k}{n\Delta} \quad k = 0, 1, \dots, \frac{n}{2}. \quad (11)$$

Finally, to facilitate the comparison and analysis of the dynamics of both the monochromatic and bichromatic modulated systems, we calculated the Poincaré sections (PS) by taking stroboscopic views of the phase plane ($\langle S_z \rangle$, $\langle \dot{S}_z \rangle$) for just the values of

$$gt = \frac{2\pi n}{\sqrt{\delta_0^2 + 4g^2(N + 1/2)}} \quad n = 0, 1, 2, \dots, \quad (12)$$

corresponding to periods equal to the Rabi period of the systems. In this equation, $\delta_0 \equiv \delta(t = 0)$.

III. RESULTS AND DISCUSSION

We solved the set of equations (8) numerically, using the implicit Runge-Kutta and semi-implicit Rosenbrock methods.¹⁶ The algorithms are adaptive (variable step-size h), and the absolute and relative tolerance errors (**atol** and **rtol**) can be varied in accordance to the pursued accuracy, and the system of differential equations. Although the initial guess for the stepsize and absolute tolerance error we used in some of the calculations were relatively small ($h \sim 10^{-7}$ and **atol** $\sim 10^{-9}$), the algorithms showed stability and reliability. The results obtained for each set of parameters used were consistent and within the expected error. However, it is necessary to mention that, for monochromatic modulations, the algorithm displayed some instability and erratic behavior when the modulated frequencies were smaller than the Rabi frequency of the system. This behavior may have to do with the very fast oscillation of the solutions when the frequency differences are nearly zero, and they are also present as denominators in the system of differential equations.

We first consider the case of no modulation of the quantized cavity field. In Fig. 1(a), we plot $\langle S_z(t) \rangle$ as a function of the scaled time gt , this is the *time series* for $\langle S_z(t) \rangle$. The parameters selected were $N = 3.50$, $g = 1.00$, $\langle S_z(0) \rangle = +1/2$ (the atom is initially in its excited state), and $\delta_0 = 0$ (there is no modulation for the cavity field). The time series is a sinusoidal curve representing how the population of the two-level atom evolves when excited with the Fock state cavity field. The mathematical behavior of $\langle S_z(t) \rangle$ was quite straightforward in this case. Figure 1(b) shows the *phase plane* plot of $\langle \dot{S}_z(t) \rangle$ against $\langle S_z(t) \rangle$. The phase trajectory is a *closed* curve with a single period of evolution. The PSD corre-

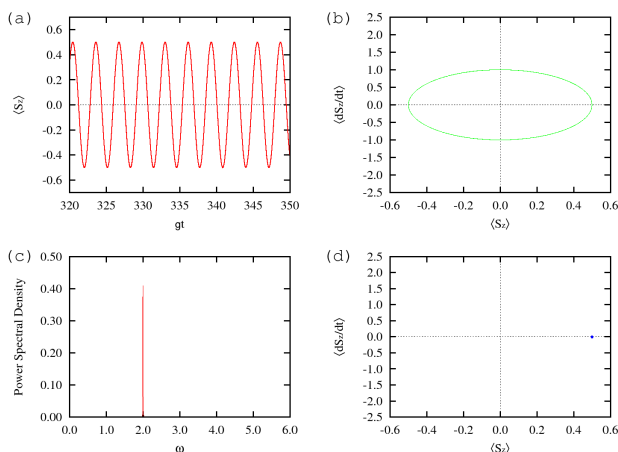


FIG. 1. Atom-cavity interaction for a fixed quantized cavity field. The parameters of the system are: $N = 3.50$, $g = 1.00$, $\delta_0 = 0$, and $\langle S_z(0) \rangle = +1/2$. The plots are: (a) time series, (b) phase plane, (c) power spectral density, and (d) Poincaré section.

sponding to the time series is shown in Fig. 1(c). We observe there is just one single frequency in the time series (Fig. 1(a)) and this concur with the result observed in Fig. 1(c), where we can see only one peak in the Fourier spectrum. This frequency matches the Rabi frequency of the system, which is equal to 4.0 in g^{-1} units. As expected, there was only a single point in the PS plot (Fig. 1(d)). Here, the PS represents when the phase trajectory crosses the time plane at every Rabi period. When we changed the parameter δ_0 to a value different from zero, all the features displayed in Fig. 1(a-d) remained unchanged, except that the Rabi frequency now turns into a more general expression given by $[\delta_0^2 + 4g^2(N + 1/2)]^{1/2}$. In addition, when we changed the value of N , the generalized Rabi frequency changed. However, the features represented in Fig. 1 still remained unchanged (single frequency in the time series, closed curve in the phase plane, and single point in the PS).

For the next set of parameters, we introduced monochromatic modulation of the cavity field in the form

$$\delta = \delta_0 \cos(\omega t).$$

We chose $N = 2.50$, $g = 1.00$, $\delta_0 = 1.00$, and $\omega = \sqrt{17}$. The corresponding Rabi frequency was $\Omega = 3.60$ (see Fig. 2(a-d)). In this case, we modulated the cavity field with a single frequency $\omega = \sqrt{17} = 4.12$, which is an *irrational* number. This fact gave rise to an interesting behavior of the dynamical evolution of the system. Figure 2(a) shows the time series for $\langle S_z(t) \rangle$ which clearly exhibits a modulation pattern. The phase plot area is almost filled by the phase trajectory from zero to the maxima of the two variables. The PSD showed two peaks, one located around Ω (the Rabi frequency of the system) and the other at ω (the modulated frequency of the cav-

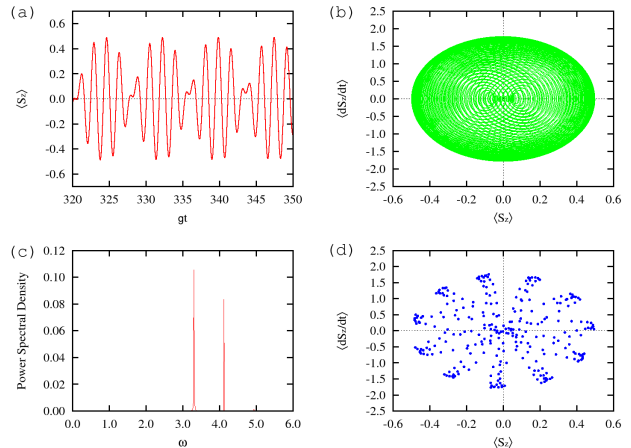


FIG. 2. Monochromatic modulation of the cavity field. The time series shows a modulation pattern. Two dominant frequencies are present in the power spectrum plot. The parameters of the system are: $N = 2.50$, $g = 1.00$, $\delta_0 = 1.00$, $\omega = \sqrt{17}$, and $\langle S_z(0) \rangle = +1/2$. The plots are: (a) time series, (b) phase plane, (c) power spectral density, and (d) Poincaré section.

ity). The PS is shown in Fig. 2(d). Although there is a lack of synchronization between the two frequencies in the system, a regular star-like pattern formed, showing that some regions of the phase plane (states of the system) were visited more often than others, suggesting the existence of possible *attractors*.

Interestingly, when $\omega = \Omega = \sqrt{17}$, the situation changed drastically as shown in Fig. 3. We achieved this condition for $N = 3.50$. The time series now shows a single frequency (see Fig. 3(a)) which is in accordance with the PSD of Fig. 3(c). The phase plot is again a closed curve as depicted in Fig. 3(b), whereas the PS shows a number of dots making up a wide single point in Fig. 3(d). These points are supposed to be at the same location; however, their separations are caused by the propagation of the computational error. In this case, when the condition $\omega = \Omega$ is satisfied, the cavity modulation of the system synchronizes with the Rabi frequency of the system, and there is a single frequency or single period of oscillation in the dynamical evolution. We also observed another intriguing situation when we chose the modulation frequency to be an overtone (integral multiple) of the Rabi frequency, i.e., $\omega = m\Omega$, ($m = 1, 2, 3, \dots$). For this situation, we observed the same synchronized behavior shown in Fig. 3. This phenomenon happens no matter the value of Ω , whether it is an integer, rational, or irrational number. When $m\Omega < \omega < (m+1)\Omega$, this synchronization is lost.

In Fig. 4, we observed another interesting behavior in the dynamics of the system. This time $\omega = (1/m)\Omega$ with $m = 100$. The ratio of ω to Ω is a fraction (with a positive integral denominator). Now the modulation frequency and the Rabi frequency are no longer synchronized with each other. Figure 4(a), displaying the time

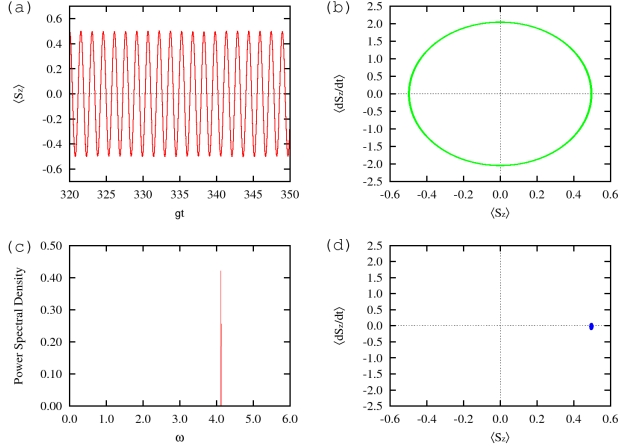


FIG. 3. The monochromatic modulation frequency of the cavity equals the Rabi frequency of the system. The parameters of the system are: $N = 3.50$, $g = 1.00$, $\delta_0 = 1.00$, $\omega = \Omega = \sqrt{17}$, and $\langle S_z(0) \rangle = +1/2$. The plots are: (a) time series, (b) phase plane, (c) power spectral density, and (d) Poincaré section. A similar situation is presented when $\omega = m\Omega$, ($m = 1, 2, 3, \dots$).

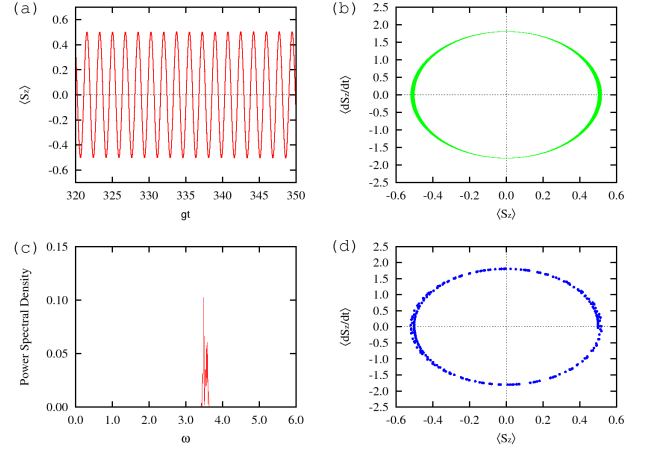


FIG. 4. Rabi frequency overtones modulating the cavity field. Quasiperiodicity is observed. The parameters of the system are: $N = 2.50$, $g = 1.00$, $\delta_0 = 1.00$, $\omega = 0.01\Omega$, and $\langle S_z(0) \rangle = +1/2$. The plots are: (a) time series, (b) phase plane, (c) power spectral density, and (d) Poincaré section.

series for $\langle S_z(t) \rangle$, apparently shows a single frequency of oscillation. However, the phase plot in Fig. 4(b) reveals a small bandwidth around the single frequency which is further confirmed in the PSD in Fig. 4(c), where there is no single frequency showing up in the spectrum. The PS for this condition is displayed in Fig. 4(d). This is typical of *quasiperiodic* motion. The phase trajectory goes around the phase plane over and over, never exactly repeating itself. As the time increases, more points will show up in the PS, but never coinciding with each other. Analyzing (3) for $\delta = \delta_0 \cos(\omega t)$, we find that the two levels in the system will cross if $\cos(\omega t) = 0$ or $t = n\pi/2\omega$ ($n = 1, 3, 5, \dots$).

Finally, we considered bichromatic modulation of the cavity field by introducing the expression

$$\delta = \delta_0 [\cos(\omega_1 t) + \cos(\omega_2 t)].$$

The parameters used were $N = 1.50$, $g = 1.00$, $\delta_0 = 1.00$, $\omega_1 = \sqrt{7}$, and $\omega_2 = \sqrt{17}$, with a Rabi frequency $\Omega = 3.00$. The atom was initially in its excited state. In Fig. 5(a), the time series of $\langle S_z(t) \rangle$ shows what can be considered a sinusoidal irregular modulation. In Fig. 5(b), the phase plot displays *quasiperiodic* behavior. This is because the modulation frequencies selected are the square root of two prime numbers, 7 and 17, and the ratio of the two frequencies ω_2/ω_1 , called the *winding number*,¹⁸ is irrational. Hence their combination produces incommensurate dynamical evolution of the physical variables $\langle S_z \rangle$ and $\langle \dot{S}_z \rangle$. The PSD corresponding to the time series is shown in Fig. 5(c). There are several frequencies in the power spectrum exhibiting the quasiperiodic behavior of the system. The PS plot (Fig. 5(d)) shows how the motion of the system never exactly repeat itself, filling the phase plane's area as time progresses.

Another interesting case of bichromatic modulation is presented in Fig. 6. For this case, the parameters used

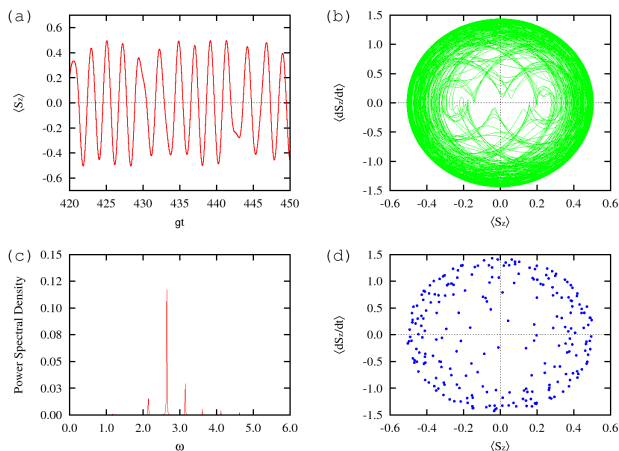


FIG. 5. Bichromatic modulation of the cavity field. The time series plot shows an irregular pattern of modulation. Different frequencies determine the dynamics of the system as shown in the power spectrum plot. The parameters of the system are: $N = 1.50$, $g = 1.00$, $\delta_0 = 1.00$, $\omega_1 = \sqrt{7}$, $\omega_2 = \sqrt{17}$, and $\langle S_z(0) \rangle = +1/2$. The plots are: (a) time series, (b) phase plane, (c) power spectral density, and (d) Poincaré section.

were $N = 1.50$, $g = 1.00$, $\delta_0 = 1.00$, $\omega_1 = \sqrt{10}$, and $\omega_2 = \sqrt{13}$, with a Rabi frequency $\Omega = 3.00$. The time series is shown in Fig. 6(a) and exhibits a similar irregular modulation pattern to that observed in Fig. 5(a). Again, this is because the ratio of the modulation frequencies is an irrational number and incommensurate dynamical evolution emerges. The phase plot in this situation brings out the quasiperiodic behavior of the system. The phase trajectory clearly outlines two distinct regions; one is an area densely populated by the states of the system, whereas the other is rarely visited by the system (see Fig. 6(b)). The PSD in Fig. 6(c) confirms this quasiperiodicity, where we observe a number of different frequencies scattered in a relatively small portion of the spectrum. Figure 6(d) shows the PS of the dynamical system. We observe how the PS points cover the area of the phase plane as time moves forward.

We also studied the behavior of the system when the two modulated frequencies ω_1 and ω_2 were overtones of the Rabi frequency Ω of the system. As it happened before with the monochromatic case, only one *dominant* frequency emerged in the PSD plot for most of the cases. Two or three marginal peaks eventually appeared in the plots, but they were negligible compared with the single dominant frequency peak. We noted a slight modulation in the time series for these cases (perhaps caused by those marginal frequencies), but still a relative single frequency could be observed. Unfortunately, the numerical algorithms were not stable and reliable enough for the chosen parameters, so we decided not to proceed further in the study of this particular case.

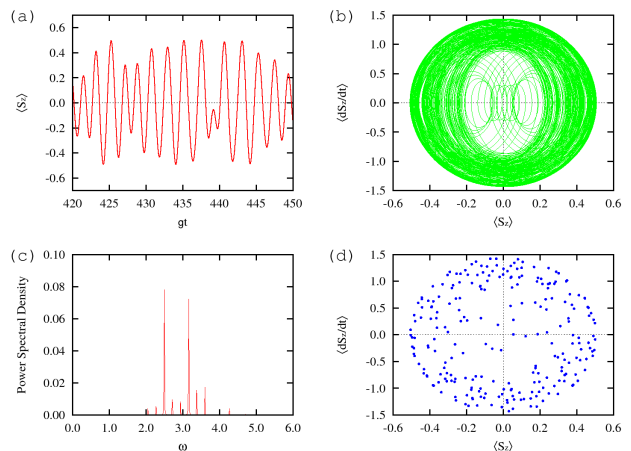


FIG. 6. Same as in Fig. 5 but now the parameters of the system are: $N = 1.50$, $g = 1.00$, $\delta_0 = 1.00$, $\omega_1 = \sqrt{10}$, $\omega_2 = \sqrt{13}$, and $\langle S_z(0) \rangle = +1/2$. The plots are: (a) time series, (b) phase plane, (c) power spectral density, and (d) Poincaré section.

IV. SUMMARY

In summary, we presented an alternative method used to study the interaction of a two-level atom with a modulated quantized field of a cavity. This method comprises different tools used to investigate nonlinear dynamical systems, like the time series, phase plane, power spectrum, and Poincaré section plots. With these instruments, we studied the dynamical evolution of a quantum system.

We described the interaction of the two-level atom with the cavity field using a Jaynes-Cumming Hamiltonian. The derived Heisenberg equations of motion of the system formed a set of three nonlinear first order differential equations. Because of the *stiffness* and complexity of the equations, we used implicit Runge-Kutta, and semi-implicit Rosenbrock numerical methods to compute the solutions. A great deal of educational value is obtained also from this part of the work, since a simple algorithm for solving ODEs cannot be used here, without compromising the accuracy and reliability of the solutions.

We explored and discussed three different scenarios: no modulation, monochromatic, and bichromatic modulation of the cavity field. For the different parameters that we used, we observed periodic and quasiperiodic behavior in the population of the atomic states. We looked specially at those cases in which the frequencies involved were integer, rational, and irrational numbers, as well as overtones of the Rabi frequency of the system. When the ratio of the frequencies was an irrational number, we observed quasiperiodic behavior of the system. For frequencies that were overtones of the Rabi frequency of the system, we noted just a single dominating frequency for the system.

The importance and final goal of this work are to help undergraduate students to understand the dynamics of a

two-level atom interacting with a quantized mode of an optical cavity, when the cavity field is modulated. These goals are achieved by working with tools used to explore nonlinear dynamical systems, and solving numerically a set of equations of motion with specialized numerical algorithms.

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